Approximating π

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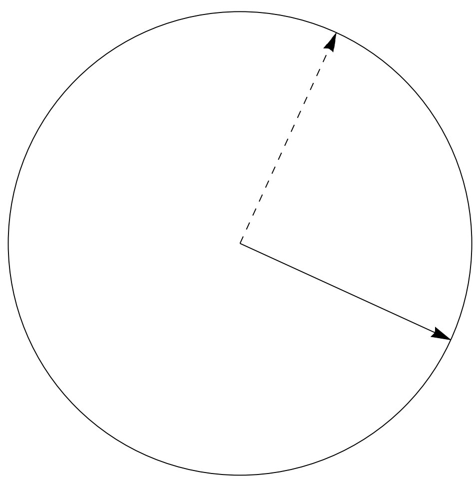
Imagine a particle travelling along a unit circle represented below. It starts from an initial position of zero and moves along the circle as represented by the solid and the dashed arrows. After finishing an entire orbit, the particle is back at its initial position of zero.



If we use the true Pi, denoted by π^* (all infinite numbers after the decimal) then after completing every $\frac{n\pi^*}{2}$ orbits, where n is an integer, the particle returns to its initial position of zero, by definition. But when we calculate π by integrating smaller chunks of motion along the orbit, then we are approximating π^*. The beauty about integration is that the motion does not even have to be perfectly circular. But lets assume that the motion is circular, but we will not use π^*.

$$y = \sqrt{1-x^2} \rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}} \rightarrow \sqrt{1+\frac{(dy)^2}{(dx)^2}} = \sqrt{\frac{1}{1-x^2}} \rightarrow \sqrt{(dx)^2+(dy)^2} = \frac{1}{\sqrt{1-x^2}}\,dx$$

The total distance travelled by the particle is described by the integral given below:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ArcSin}[x]$$

$$\text{ArcSin}[1/2] - \text{ArcSin}[0] = \frac{\pi}{6} - 0 \rightarrow \pi = 6\,\text{ArcSin}[1/2]$$

Since we want to estimate π, we will have to use a Taylor series to approximate ArcSin[x]. The series converges quickly if we use (π/6), rather than just π. So we need to multiply the Taylor series for ArcSin[x] by 6 :

5th order series (s5) = $6x + x^3 + \frac{9x^5}{20}$

20th order series (s20) = $6\,x + x^3 + \frac{9\,x^5}{20} + \frac{15\,x^7}{56} + \frac{35\,x^9}{192} + \frac{189\,x^{11}}{1408} + \frac{693\,x^{13}}{6656} + \frac{429\,x^{15}}{5120} + \frac{19\,305\,x^{17}}{278\,528} + \frac{36\,465\,x^{19}}{622\,592}$

50th order series (s50) =

$$6\,x + x^3 + \frac{9\,x^5}{20} + \frac{15\,x^7}{56} + \frac{35\,x^9}{192} + \frac{189\,x^{11}}{1408} + \frac{693\,x^{13}}{6656} + \frac{429\,x^{15}}{5120} + \frac{19\,305\,x^{17}}{278\,528} + \frac{36\,465\,x^{19}}{622\,592} + \frac{46\,189\,x^{21}}{917\,504} + \frac{264\,537\,x^{23}}{6\,029\,312} + \frac{2\,028\,117\,x^{25}}{52\,428\,800} + \frac{1\,300\,075\,x^{27}}{37\,748\,736} + \frac{15\,043\,725\,x^{29}}{486\,539\,264} + \frac{29\,084\,535\,x^{31}}{1\,040\,187\,392} + \frac{300\,540\,195\,x^{33}}{11\,811\,160\,064} + \frac{350\,040\,933\,x^{35}}{15\,032\,385\,536} + \frac{6\,806\,351\,475\,x^{37}}{317\,827\,579\,904} + \frac{4\,418\,157\,975\,x^{39}}{223\,338\,299\,392} + \frac{103\,384\,896\,615\,x^{41}}{5\,634\,997\,092\,352} + \frac{201\,846\,702\,915\,x^{43}}{11\,819\,749\,998\,592} + \frac{17\,534\,158\,031\,x^{45}}{1\,099\,511\,627\,776} + \frac{1\,543\,768\,261\,425\,x^{47}}{103\,354\,093\,010\,944} + \frac{24\,185\,702\,762\,325\,x^{49}}{1\,724\,034\,232\,352\,768}$$

As we increase the terms, we get closer and closer to approximating the true value of π. Here are the values of π estimated using each series evaluated to upto sixteen decimals:

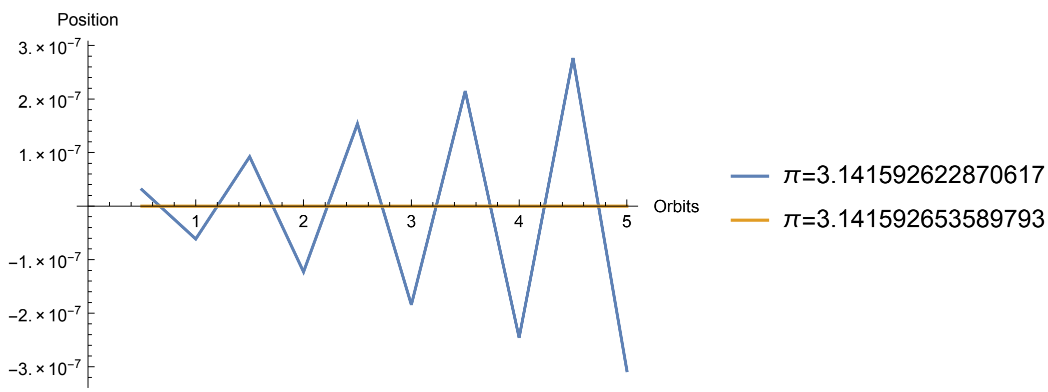
$\pi(5) \approx 3.1390625$

$\pi(20) \approx 3.141592622870617$

$\pi(50) \approx 3.141592653589793$

The position of the particle changes according to a Sine function:

Orbit #	π(5)=3.1390625	π(20)=3.141592622870617	π(50)=3.141592653589793
0.5	0.00253015	3.07192×10^{-8}	1.22465×10^{-16}
1.	-0.00506029	-6.14384×10^{-8}	-2.44929×10^{-16}
1.5	0.00759039	9.21575×10^{-8}	3.67394×10^{-16}
2.	-0.0101204	-1.22877×10^{-7}	-4.89859×10^{-16}
2.5	0.0126504	1.53596×10^{-7}	6.12323×10^{-16}
3.	-0.0151803	-1.84315×10^{-7}	-7.34788×10^{-16}
3.5	0.0177101	2.15034×10^{-7}	8.57253×10^{-16}
4.	-0.0202398	-2.45753×10^{-7}	-9.79717×10^{-16}
4.5	0.0227694	2.76473×10^{-7}	1.10218×10^{-15}
5.	-0.0252988	-3.07192×10^{-7}	-1.22465×10^{-15}



An irrational number such as π has to be approximated because no computer in the world has access to infinite processing power to print a number with infinite decimal places. This implies that approximating the value of π leads to error in the measurement of the position of the particle, save for the very rare instances where the particle is travelling along an ideal perfect circular path. This means that our model for motion implies that nature is unpredictable, because our initial position of the particle is not known accurately.